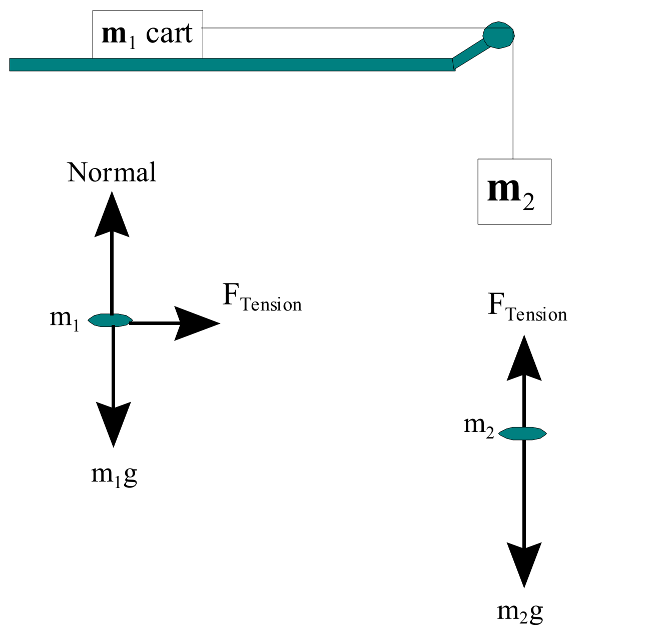
Rotational Inertia

Consider a cart on four wheels (mass m_1) on a level track connected by a light string over a pulley to a second suspended mass m_2 (see diagram below). Assume both cart and pulley are frictionless, and ignore the mass of the string. Assume the string is non-stretchable, then both masses have the same acceleration. The tension force $F_{tension}$ in the string is the same on both sides of the pulley, because we are neglecting the friction in the pulley for the moment. Note: the suspended weight m_2g is the ***applied force*** causing the entire system to accelerate. Analyze the motion of this system from the point of view using: (a) forces and (b) energy.

A. **Force** analysis of the motion of the system (cart plus suspended mass).

The free-body diagrams for the cart and suspended mass are illustrated to assist in analysis.



Cart	**Suspended Mass**
$F_{net} = F_{tension}$	$F_{net} = m_2g - F_{tension}$
$m_1a_1 = F_{tension}$	$m_2a_2 = m_2g - F_{tension}$

Combining these two equations by substituting $F_{tension}$ and accounting for the friction, results in:

$$m_2 g - F_{friction} = \left(m_1 + m_2\right) a \quad \textbf{(1)}$$

Note, Equation (1) states: the **net** force (applied force m_2g less the opposing friction) and the acceleration $\boldsymbol{a}$ of the system are proportional with combined mass constant of $(m_1 + m_2)$. We may view the combined cart-plus-suspended mass as one system of total mass $m_1 + m_2$. In this analysis, the cause of any rotation of the pulley and the cart wheels has been ignored.

B. **Energy** analysis of the motion of the system (cart plus suspended mass).

As the system moves it gains kinetic energy of motion at the expense of the potential energy of position of the suspended weight m_2g. When the suspended weight falls through vertical distance *y* its potential energy decreases by m_2gy, and using the **conservation of energy**:

$$m_2 g\, y = \quad KE^{cart}_{translation} \quad + \quad KE^{pulley}_{rotation} \quad + \quad KE^{wheels}_{rotation} + \quad IE_{friction}$$

$$m_2 g\, y = \frac{1}{2}\left(m_1 + m_2\right) v^2 + \frac{1}{2} I_{pulley}\omega^2 + \frac{1}{2} I_{wheels}\omega^2 + RF_{friction}\theta$$

R = radius of pulley or wheels.
For non-slip rotational motion: $\omega = \dfrac{v}{R}$ *and thus* $I\omega^2 = \dfrac{I}{R^2} v^2$

$$m_2 g\, y = \frac{1}{2}\left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{pulley} + 4\left(\frac{I}{R^2}\right)_{wheel} \right] v^2 + RF_{friction}\theta$$

We differentiate this energy equation with respect to time and recognize that:

$$\dot{y} = \frac{dy}{dt} = system\ velocity = v$$

$$\dot{v} = \frac{dv}{dt} = system\ acceleration = a$$

$$\dot{\theta} = \frac{d\theta}{dt} = rotational\ velocity = \omega = \frac{v}{R}$$

$$m_2 g\, \dot{y} = \frac{1}{2}\left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{pulley} + 4\left(\frac{I}{R^2}\right)_{wheel} \right] 2\, v\, \dot{v} + RF_{friction}\dot{\theta}$$

$$m_2 g \; = \; \left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{pulley} + 4\left(\frac{I}{R^2}\right)_{wheel} \right] a \; + \; F_{friction} \qquad \textbf{(2)}$$

Note: Equation (**2**) is the same as (**1**) stating force is proportional to acceleration but with a different mass constant which incorporates the ***rotational*** inertia of the pulley and wheels.

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