Harmonic Motion (HM)

Oscillation with Laminar Damping

If you don't know the units of a quantity you probably don't understand its physical significance.

Simple HM

Hooke's Law: $\vec{F} = -k\,\vec{x}$ definitions: $f\,T \equiv 1$ $\omega \equiv 2\pi f$

$x = A \sin(\phi_o + \omega t)$ $\boldsymbol{\omega^2 = k/m}$ or $(2\pi/T)^2 = k/m$

$v = A\,\omega \cos(\phi_o + \omega t)$ Simple pendulum: $k = mg/L$ $\omega^2 = g/L$

$a = -A\,\omega^2 \sin(\phi_o + \omega t)$ Cork on water: $k = \rho_{water} A g$ $\omega^2 = \rho_{water}\, g/\rho_{cork} L$

kinetic energy = $\tfrac{1}{2}mv^2$

elastic potential energy = $\tfrac{1}{2}kx^2$

conservation of mechanical energy: $KE_f + PE_f = KE_i + PE_i$

if friction can be neglected ($W_{non\text{-}conservative} = 0$)

Damped HM $\omega_o^2 \equiv k/m$

damping force $F_{damp} = -\,\mathbf{c}\mathbf{v}$ laminar damping force proportional to the velocity **v**

energy: $E = E_o\, e^{-\frac{c}{m}t}$ $\dfrac{1}{\tau} = \dfrac{c}{m}$ *for laminar damping*

quality: $Q \equiv \dfrac{E/T}{P_{ave}/2\pi}$ *or* $\dfrac{\Delta E}{E} = -\dfrac{\Delta t}{Q/\omega}$ $\dfrac{Q}{\omega} \equiv \tau$ *decay constant*

a) low damping (under damped): $c^2 < 4mk$

position: $x = A_o\, e^{-\frac{c}{2m}t} \sin(\omega t + \varphi_o)$ $\omega = \sqrt{\left(\dfrac{k}{m}\right) - \left(\dfrac{c}{2m}\right)^2}$

Amplitude: $A = A_o\, e^{-\beta t}$ $\beta = \dfrac{c}{2m}$ *for laminar damping*

Damping decreases the frequency of oscillation, and also the amplitude as time increases.

b) medium damping (critical damping): $c^2 = 4mk$
The system returns to equilibrium in the shortest time without oscillating.

c) large damping (over damped): $c^2 > 4mk$
The system takes a longer time to reach equilibrium without oscillating.

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Forced HM and Resonance

Under the influence of an external driving force $\mathbf{F} = \mathbf{F_o} \sin \omega t$ how will the system respond?

Position: $\mathbf{x = A \sin(\omega t + \phi)}$ displacement always lags ($\phi < 0$) the driving force

$$Amplitude\ A = \frac{F_o / m}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega}{\tau}\right)^2}} \qquad \tan\varphi = -\frac{\omega/\tau}{\omega_o^2 - \omega^2}$$

a) At low driving frequency $\omega << \omega_o$
Response controlled by spring (k)

$$A = \frac{F_o / m}{\omega_0^2} = \frac{F_o}{k} \qquad \varphi \to 0$$

b) At resonance driving frequency $\omega = \omega_o$
Response controlled by damping (c)

$$A = \frac{F_o / m}{\omega_o / \tau} = \frac{F_o / \omega_o}{c} \qquad \varphi \to -\frac{\pi}{2}$$

c) At high driving frequency $\omega >> \omega_o$
Response controlled by inertial mass (m)

$$A = \frac{F_o / m}{\omega^2} = \frac{F_o / \omega^2}{m} \qquad \varphi \to -\pi$$

$$\frac{response\ at\ resonance}{response\ at\ zero\ frequency} = \frac{A(\omega = \omega_o)}{A(\omega = 0)} = \frac{\tau / \omega_o}{1/\omega_o^2} = \omega_o\, \tau = Q$$

Power Absorption:

Power is the time average of the force times the velocity (see exercise 2):

$$Power = \langle F \cdot v \rangle = P_{res} \frac{(\omega/\tau)^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega/\tau)^2} \qquad where \quad P_{res} = \frac{1}{2} m \left(\frac{F_0}{m}\right)^2 \tau$$

Near resonance ($\omega \approx \omega_o$) the half-power points occur when: $\omega_o^2 - \omega^2 = \omega/\tau$
or $2\,\omega_o\, \Delta\omega \approx \omega/\tau$

Thus the full width ($2\,\Delta\omega$) of the resonance at half maximum power is equal to $1/\tau$ and the quality of an oscillator ***Q*** measures the sharpness of damping near resonance:

$$Quality \quad Q = \omega_o\, \tau = \frac{\omega_o}{2\Delta\omega} = \frac{frequency\ at\ resonance}{full\ width\ at\ half\text{-}maximum\ power}$$

Numerical Example of Harmonic Oscillator

Given oscillator data:

	mass	m = 1 gram
	spring constant	k = 10 N/m
	relaxation time	τ = 0.5 second

Then:

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{10\,\frac{N}{m}}{10^{-3}\ kg}} = 10^2\ \frac{radian}{second} = 16\ \frac{cycles}{second}$$

$$free\ oscillation\ frequency \quad \omega = \sqrt{\omega_o^2 - \left(\frac{1}{2\,\tau}\right)^2} = \sqrt{10^4 - 1} = 10^2\ \frac{radian}{second}$$

$$Quality\ factor \quad Q = \omega_o\ \tau = 10^2\ \frac{radian}{second}\ 0.5\ second = 50$$

The time for the amplitude to damp to e^{-1} of its initial value is: $\frac{1}{\beta} = 2\,\frac{m}{c} = 2\ \tau = 2\,(0.5\ \sec) = 1\ \sec$

$$the\ value\ of\ the\ damping\ constant\ is: \quad c = \frac{m}{\tau} = \frac{1\ gram}{0.5\ second} = 2\ \frac{gram}{second}$$

Given a driving force: F = 0.1 N sin (90 t) thus F_o = 0.1 N ;

$\frac{F_o}{m} = \frac{0.1\ N}{1\ gram} = 0.1\ \frac{N}{gram} = 100\frac{N}{kg}$ and the driving frequency is ω = 90 rad/sec

The response of the oscillator is: $\varphi = -5.4°$

$$Amplitude\ A = \frac{100\ N\ /\ kg}{\sqrt{\left(10^4 - 90^2\right)^2 + \left(\frac{90}{0.5}\right)^2}} = \frac{100}{1909} = 5\ cm \qquad \tan\ \varphi = \frac{90\ /\ 0.5}{90^2 - 10^4} = -0.095$$

$$the\ amplitude\ response\ at\ resonance\ is\ A(\omega = \omega_o) = \frac{F_o\ /\ m}{\omega_o\ /\ \tau} = \frac{100\ N\ /\ kg}{\frac{100\ rad\ /\ \sec}{0.5\ second}} = 50\ cm$$

$$the\ amplitude\ at\ low\ frequency\ is\ A(\omega = 0) = \frac{F_o}{k} = \frac{0.1\ N}{10\ N\ /\ m} = 1\ cm$$

The power absorption at resonance: $P_{res} = \frac{1}{2}\,10^{-3}\,kg\,(100\ N\ /\ kg)^2\ 0.5\ \sec = 2.5\ Watt$

The full width of the resonance curve between half-power points is:

$2\ \Delta\omega = \omega_o/Q$ = 100 rad/sec / 50 = 2 rad/sec.

Exercise 1

Show **x=Asin(ωt+ϕ)** satisfies the equation of motion of a ***driven*** harmonic oscillator.

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F_o \sin\omega t \qquad \frac{c}{m} \equiv \frac{1}{\tau} \qquad \frac{k}{m} \equiv \omega_o^2$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_o \sin\omega t}{m}$$

Substituting the first and second differentiation of the proposed solution x=Asin(ωt+ϕ):

$$\left(\omega_o^2 - \omega^2\right) A\sin\left(\omega t + \varphi\right) + \frac{\omega}{\tau} A\cos\left(\omega t + \varphi\right) = \frac{F_o}{m}\sin\left(\omega t\right)$$

using: sin(ωt+ϕ) = sinωt cosϕ + cosωt sinϕ
cos(ωt+ϕ) = cosωt cosϕ - sinωt sinϕ

$$\left[\left(\omega_o^2 - \omega^2\right)\cos\varphi - \frac{\omega}{\tau}\sin\varphi\right] A\sin\omega t + \left[\left(\omega_o^2 - \omega^2\right)\sin\varphi + \frac{\omega}{\tau}\cos\varphi\right] A\cos\omega t = \frac{F_o}{m}\sin\omega t$$

This equation can only be satisfied for all time *t* if the coefficients of sinωt and cosωt are both zero, which establishes A and the phase angle ϕ:

$$A = \frac{F_o / m}{\left(\omega_o^2 - \omega^2\right)\cos\varphi - \frac{\omega}{\tau}\sin\varphi} = \frac{F_o / m}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega}{\tau}\right)^2}}$$

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = -\frac{\omega / \tau}{\omega_o^2 - \omega^2}$$

where we have used $\sin^2\phi + \cos^2\phi \equiv 1$ in order to obtain:

$$\sin\varphi = \frac{-\omega / \tau}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega}{\tau}\right)^2}} \qquad \cos\varphi = \frac{\omega_o^2 - \omega^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega}{\tau}\right)^2}}$$

Exercise 2

Derive the power absorption P(ω) for a driven harmonic oscillator.

For the motion of a forced or driven oscillator we have:

force $\mathbf{F = F_o \sin \omega t}$; position $\mathbf{x = A \sin(\omega t + \phi)}$; velocity $\mathbf{v = \omega A \cos(\omega t + \phi)}$

Note the phase angle ϕ of the position ***x*** relative to the driving force ***F*** and an additional 90° for the velocity ***v*** which is the derivative of the position.

The time averaged power is the time average of the force times the velocity

Power = ⟨ $F_{external}$ · velocity ⟩ ⟨⟩ ≡ **time average**

Power = ⟨ (F_o sinωt) · { ωA cos(ωt + ϕ) } ⟩

cos(ωt + ϕ) = cosωt cosϕ - sinωt sinϕ

Power = F_o ω A ⟨ sinωt cosωt cosϕ − sinωt sinωt sinϕ ⟩

The time average of sinωt cosωt is zero, that is ⟨ sinωt cosωt ⟩ = 0
and the time average of $\sin^2\omega t$ is one half, that is ⟨ $\sin^2\omega t$ ⟩ = ½

Power = F_o ω A (0 − ½ sinϕ)

From exercise 1: $A = \dfrac{F_o / m}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2}}$ $\sin\varphi = \dfrac{-\omega / \tau}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2}}$

$$Power = \frac{1}{2} F_o \; \omega \frac{F_o}{m} \frac{\omega / \tau}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2}$$

$$Power = \frac{1}{2} m \left(\frac{F_o}{m}\right)^2 \tau \frac{(\omega / \tau)^2}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2} = \frac{\frac{1}{2} m \, (\omega A)^2}{\tau}$$

Note the frequency dependence factor
containing ω, damping τ=c/m, spring ω_o^2= k/m

$$factor = \frac{(\omega / \tau)^2}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2}$$

Exercise 3

Consider a series electromagnetic circuit consisting of: an inductor L, a resistor R, a capacitor C, and an applied power supply $\mathcal{E} = \mathcal{E}_o \sin \omega t$. Show that such a circuit behaves identically as a harmonic oscillator.

Applying Kirchhoff's Voltage Law to the series circuit loop, we obtain:

$$L\frac{\Delta i}{\Delta t} + Ri + \frac{1}{C}q = E_o \sin(\omega t)$$

where ***q*** is the time-varying **q**uantity of electric charge stored in the capacitor and ***i*** is the time varying **i**ntensity of current as the charge oscillates through the series circuit.

Substituting for ***i*** ≡ Δq/Δt :

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_o \sin(\omega t)$$

$$m\ddot{x} + c\dot{x} + k\,x = F_o \sin(\omega t)$$

The form of the electric circuit equation is identical to that of the mechanical, laminar-damped, driven forced oscillator. Mathematically, the two equations describe the same behavior. It remains to identify the quantities that correspond to each other and identify their physical significance and the role each plays.

Inductance	L	⟸⟹	m	mass
Resistance	R	⟸⟹	c	damping factor
elastance	1/C	⟸⟹	k	spring's elastic constant
$\mathcal{E}$lectro Motive Force	E_o	⟸⟹	F_o	maximum strength of driving force

Thus:
Inductance plays the role of inertia, as expected since it takes time to establish a current in an inductor when a voltage is applied to it.

Resistance plays the role of damping, as expected because it absorbs energy similar to a dashpot or a shock absorber in a mechanical system.

The inverse of capacitance plays the role of elasticity, as expected because a capacitor stores electric charge and energy similar to the elastic energy storage in a spring.
The natural frequency is therefore $\omega_o^2 = \frac{elasticity}{inertia} = \frac{1/C}{L} = \frac{1}{LC}$
which is precisely the resonance frequency of the so-called electrical "tuning" circuit.

The phase angle between the applied voltage and the ***current*** is $\tan\varphi = \frac{\omega L - 1/\omega C}{R}$

At resonance the current and voltage are in phase $\phi = 0$, the circuit acts as a pure resistance, and for a given impressed $\mathcal{E}$mf the current in the circuit is a maximum.
The resonance full width at half-maximum power points is $(2\Delta\omega) = 1/\tau = R/L$.
The Quality factor of the electromagnetic oscillator is $Q = \omega_o \tau = \omega_o L/R$ etc.

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Exercise 4

Consider the damping factor for a flat plate moving normal to its plane with speed ***v*** through a low-pressure, rarefied gas. Assume the mean free path of the gas molecules is large compared to the dimensions of the plate so that the deceleration of the plate may be thought of in terms of individual collisions between the plate and the molecules rather than hydrodynamic flow.

The drag force on the plate will be proportional to the rate at which molecules strike the plate multiplied by the average momentum transfer per molecule.

The rate at which molecules strike the plate is proportional to the relative velocity.
Consider the relative velocity between the incoming molecules and the plate.
Suppose the molecules move in only one direction with speed v_m or with rms speed.
On one side of the plate the relative velocity is $\mathbf{v_m + v}$; on the other side it is $\mathbf{v_m - v}$.

The average momentum transfer is itself proportional to the relative velocity.
The net drag force on the plate opposing the motion will thus be proportional to:

$$F_{damping} = -\left|F_{forward} - F_{backward}\right| \quad \propto - \left| (v_m - v)^2 - (v_m + v)^2 \right|$$

Case: v << molecular velocity

$$F_{damping} \propto -4\, v_m\, v \propto -v$$

The damping is proportional to the first power of the velocity.
For viscous hydrodynamic flow this is referred to as ***laminar*** flow.

Case: v >> molecular velocity

In this case there is no forward force as the molecules can't catch up and the molecular velocity is negligibly small and ignored.

$$F_{damping} \propto -v^2$$

The damping is proportional to the second power of the velocity.
For viscous hydrodynamic flow this is referred to as ***turbulent*** flow.

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