Joule and the Conservation of Energy

The unit Joule, abbreviated J, is shorthand for Newton•meter (N•m)
which is defined via the definition of work:

Work $\equiv$ || Force || **(cos θ)** || Displacement ||
where θ is the angle between the force vector and the displacement vector.

Note that multiplying two vectors in this manner results in a ***scalar*** number or value. It is independent of the vectors' orientation in space. It is the product of: the component of one vector along the direction of the other vector, multiplied by the magnitude of the other vector. The ± sign of work is determined by (cos θ) since the magnitude of a vector, || ||, is always positive. Work is negative when the angle θ lies between 90° and 270°; a prime example is a friction force which always opposes the motion (θ = 180°). When force and displacement are perpendicular to each other (θ=90°) the work done is 0.

Whenever work is done on an object, then one or more of the object's motional, positional, or internal amount of joules change; that is in general an object can have:

Joules of motion	called ***kinetic*** energy
Joules of position	called ***potential*** energy
Joules internal to the object	called ***internal*** energy.

Now there is a law of nature, discovered about 1840 and always observed to be true since then, which states that energy is conserved. Joules cannot be created or destroyed, they can only be transferred or transformed; it is called the principle of conservation of energy. This law of energy conservation is connected to the time symmetry in nature. No exceptions to this law have ever been encountered; it is thought ***always*** to be true.

example: Suppose you observe an object undergoing some process in nature.
You notice that ***before*** the process (initially) the object has:

motional joules	Kinetic Energy	$KE_{initial} = 21$ J
positional joules	Potential Energy	$PE_{initial} = 15$ J
internal joules	Internal Energy	$IE_{initial} = 42$ J

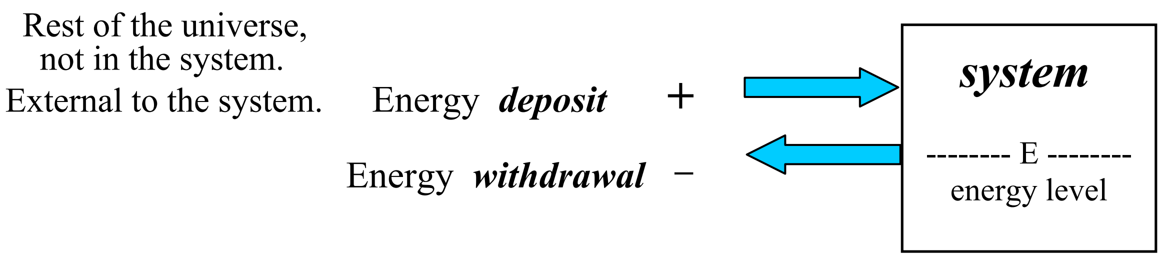
During this particular process the internal energy of the object does not change and the object's positional energy reduces by 7 J.
What is the ***final*** motional or kinetic energy of the object?

solution: we write the conservation of energy:
energy ***before*** the process = energy **after** the process — Answer:
$21 + 15 + 42\ \text{J} = KE_{final} + (15-7) + 42\ \text{J}$ — $KE_{final} = 28$ J

In this example, we have assumed that the object was left to itself and was not given any additional energy nor did it loose any energy; the object was treated as an ***isolated*** object.

General - Concept of System

In general, consider a ***system*** instead of only an object; it is an imaginary box or sphere boundary, around or containing the object or region of interest. Anything outside this box or sphere does ***not*** belong to the system of interest; it belongs to the rest of the universe. It is essential there be no ambiguity about what belongs to a system and what does not. We keep track of the amount of energy (Joules) entering (+) or leaving (−) the system. We speak of the system having an ***energy level E*** analogous to the water level in a well.



The most general form of the conservation of energy law is: (system isolated or not)

$$\mathbf{J}_{\text{in or out}} = \boxed{\textbf{Change in energy of the system } \Delta E}$$

Note:

1. Energy is thus involved in nature's ***changes*** during processes and it tracks Joules.
2. It is the ***net*** amount of energy entering and leaving the system or transferred into the system from the rest of the universe that determines how much the energy level of the system changes. We denote this net amount of energy here by $J_{\text{in or out}}$. When the energy ***enters*** the system $J_{\text{in or out}}$ is a positive number, ΔE is positive and the system's energy level ***rises***. When energy ***leaves*** the system $J_{\text{in or out}}$ is a negative number, ΔE is negative and the system's energy level ***decreases***.
3. For a system isolated from the rest of the universe (no interaction) $J_{\text{in or out}}$ is zero.
4. Work is a measure of the energy transferred by a force acting over a distance.

example:

Suppose you have a system whose initial energy level before some process is 78 J and during the process 14 J ***enter*** the system from the rest of the universe, for example electromagnetic light photons or sound or heat enters the system. What is the final energy level of the system after the process?

solution: We write the conservation of energy law: $+14\text{ J} = E_{\text{final}} - E_{\text{initial}}$

$$+14\text{ J} = E_{\text{final}} - 78\text{ J}$$

Answer:

$E_{\text{final}} = 92$ J ; the energy level of the system increased by 14 J from 78 J to 92 J.

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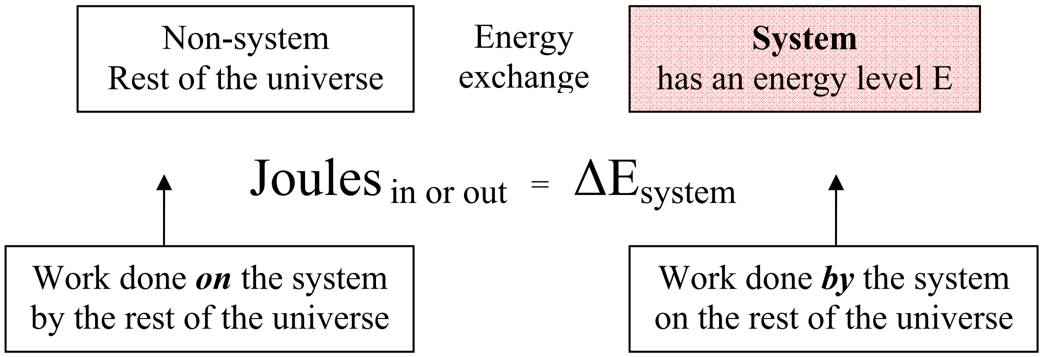
Work and Force: Conservative & Non-Conservative Forces

Work is a measure of the energy transferred by force acting over a distance.
Identify some system:



$$\text{Joules}_{\text{in or out}} = \Delta E_{\text{system}}$$

For work done by forces within a system: $0 = \Delta E + \text{Work}$

Note:
Work done ***by*** a system is at the expense of the system's energy; i.e. the work done by a system always has the opposite sign (±) of the change in energy of the system.

A. *Conservative forces*

Consider a system of two objects and their interactive conservative force. For example, a mass m free falls toward a second fixed mass M. Conservative forces change the ***potential*** energy (PE) of a system, which can convert it back if the process is reversed.

$0 = \Delta PE + \text{Work}$
$0 = \Delta PE + F_{cons} (\cos 0°)\, \Delta s$ force is parallel to and in the direction of the displacement Δs
$0 = m\, \Delta P + m\, \text{Field}\, (1)\, \Delta s$ where ΔP is the change in the potential.

For conservative forces the ***field*** is the negative gradient of the potential: $Field = -\dfrac{\Delta P}{\Delta s}$

B. *Non-Conservative forces*

Consider a system of two objects and an interactive non-conservative force. For example, a mass m slides with friction on a second mass M. Non-Conservative forces change the ***internal*** energy (IE) of a system, which can ***not*** convert it back if the process is reversed.

$0 = \Delta IE + \text{Work}$
$0 = \Delta IE + F_{non\text{-}cons} (\cos 180°)\, \Delta s$ friction opposes the motion: $\cos 180° = -1$
$\Delta IE = -\text{Work}_{non\text{-}cons} = -F_{non\text{-}cons} (-1)\, \Delta s$ = a positive number > 0

Non-Conservative forces cause the internal energy of a system to ***increase***.

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Example: Conservative Force

A particle has mass 2.8 x 10^{-6} kg and carries electric charge –2.5 μCoulomb. Upon release, in a region having electric field, this particle is observed to have speed 45 m/s at point ***A***, and later arrives at point ***B*** with speed 25 m/s.

a) ¿What is the potential difference (voltage drop) between the two points ?

Consider the particle and the region with the electric field as the system. After release no energy is put into the system; it is an isolated system:

$$0 = \Delta PE + \Delta KE$$

$$0 = q\Delta V + \frac{1}{2}m\left(v_f^2 - v_i^2\right) \qquad q = \text{charge} \qquad \text{m=mass}$$

$$\Delta V = -\frac{\frac{1}{2}m\left(v_f^2 - v_i^2\right)}{q} = \frac{\frac{1}{2}\,2.8x10^{-6}\,kg\,\left(25^2 - 45^2\right)\frac{m^2}{s^2}}{-2.5x10^{-6}\ Coulomb} = -784\ \frac{Joule}{Coulomb}$$

b) ¿Which of the two points ***A*** or ***B*** is at the higher electric potential ?

Point ***A*** is the initial point and point ***B*** is the final point. Thus:

$$\Delta V \equiv V_{final} - V_{initial} = V_{\mathbf{B}} - V_{\mathbf{A}} = -784 \text{ volt}$$

$$V_{\mathbf{A}} = V_{\mathbf{B}} + 784 \text{ volt}$$

Answer: Point ***A*** has higher electric potential than point ***B***.

c) ¿What is the direction of the electric field between the two points ***A*** and ***B*** ?

Work ***on*** the particle by the field is at expense of the change in potential energy:

$$\Delta PE = -\|\, F_{conservative} \,\|\ \cos\theta\ \|\Delta s\|$$

$$q\Delta V = -\|\ \overrightarrow{qE}_{field}\ \|\ \cos\theta\ \|\Delta s\|$$

$$q\left(-784\,volt\right) = -\|\ \overrightarrow{qE}_{field}\ \|\ \cos\theta\ \|\Delta s\| \geq 0$$

Hence ($\cos\theta \leq 0$) the *force* is opposite in direction to the displacement Δs, but with negative charge the ***field*** is in the direction of the displacement, from point ***A*** to point ***B***; it decelerates the negatively-charged particle.

Example: Non-Conservative Force

A child on a sled slides *down* a 12° hill without changing its speed.
The total mass of the sled and the child is 45 kg; the vertical *drop* is 42 m.
Note: the *length* of the hill is Δs = Δh/sin*12*° = 42 m/0.21 = 200 m.

a) ¿ What is the change in internal energy of the system ?

Consider the sled/child and the hill with the gravity field as the system.
After release no energy is put into the system; it is an isolated system:

$$0 = \Delta PE + \Delta KE + \Delta IE \qquad with \;\; \Delta KE = 0$$

$$\Delta IE = -\Delta PE = -\left(-(45 kg)(9.81 \frac{N}{kg})(42\, m) \right) = +18540\, J$$

$$when\ the\ potential\ energy\ of\ the\ system\ decreases.$$

Answer: the internal energy of the system ***in***creased by 19 kJ.
This energy comes from the ***de***crease in gravitational potential energy.
There was ***no*** change in the translational kinetic energy KE of the system.

b) ¿ What frictional force acts between the sled and the hill ?

Consider the work done on the sled by the system's friction force.

$$0 = \Delta IE + Work_{non-conservative}$$

$$\Delta IE = -Work_{non-conservative}$$

$$\Delta IE = -\|\vec{F}_{friction}\| \cos\theta \, \|\Delta s\| \geq 0$$

$$note:$$

$$\cos\theta \leq 0 \;\; or\ with\ the\ friction\ force\ along\ the\ incline \;\; \cos 180° = -1$$

$$\|\vec{F}_{friction}\| = \frac{\Delta IE}{-(\cos 180°)\Delta s} = \frac{+18540\, J}{-(-1)\, 200\, m} = +93\, N$$

Answer: the magnitude of the friction force is 93 N;
its direction is opposite the displacement vector Δs, thus uphill.

Notation and Example

Starting with $\mathbf{J}_{\text{in or out}} = \Delta E_{\text{system}}$

we have $E_{\text{final}} = E_{\text{initial}} + \mathbf{J}_{\text{in or out}}$

or $KE_{\text{final}} + PE_{\text{final}} + IE_{\text{final}} = KE_{\text{initial}} + PE_{\text{initial}} + IE_{\text{initial}} + \mathbf{J}_{\text{in or out}}$

or $KE_{\text{final}} - KE_{\text{initial}} + PE_{\text{final}} - PE_{\text{initial}} + IE_{\text{finall}} - IE_{\text{initial}} = \mathbf{J}_{\text{in or out}}$

or $\Delta KE + \Delta PE + \Delta IE = \mathbf{J}_{\text{in or out}}$

where we have used the concept that the total energy of the system is made up of the system's kinetic plus potential plus internal energy. Often we have a process during which the internal energy of the system does not change, we then have ΔIE is zero and we speak of the mechanical energy **ME** or its change; in such a case we have

or $\Delta KE + \Delta PE + \text{zero} = \mathbf{J}_{\text{in or out}}$

or $\Delta ME + \text{zero} = \mathbf{J}_{\text{in or out}}$

where *mechanical energy* is defined as the sum of the kinetic and the potential energy.

example: At a carnival, you manage to just barely ring the bell by striking a target with a 7-kg hammer which sends a 0.3-kg metal piece upward toward the 4.75-m high bell. With what speed must the hammer be moving when it strikes the target, if ¾ or 75 % of its kinetic energy is dissipated into heat, friction, and sound, leaving ¼ or 25 % to do the mechanical work?

solution: Choose the system of interest to be the metal piece (m=0.3 kg) and the hammer transfers energy into the system. The metal piece is at rest initially and also as it just strikes the bell, so the change in its kinetic energy is zero. The input joules are the kinetic energy of the hammer which is $\frac{1}{2} M V^2$ as the hammer comes to rest.

We write the conservation of energy:

$$\mathbf{J}_{\text{in or out}} = \Delta KE + \Delta PE + \Delta IE$$

$$\tfrac{1}{2} M V^2 = \text{zero} + mg\Delta h + \tfrac{3}{4}\left(\tfrac{1}{2} M V^2\right)$$

or $$\tfrac{1}{4}\left(\tfrac{1}{2} M V^2\right) = \text{zero} + mg\Delta h$$

solve for V when $M = 7$ kg, $m = 0.3$ kg, $g = 9.81\ \text{m/s}^2$ $\Delta h = 4.75$ m

Answer: $V = 4$ m/s

Energy Expressions

1. Kinetic Energy

$$translational = \frac{1}{2} mass\ (\ velocity\ )^2$$

$$rotational = \frac{1}{2}\begin{pmatrix} rotational \\ inertia \end{pmatrix}\begin{pmatrix} rotational \\ velocity \end{pmatrix}^2$$

$$relativistic = Total\ Energy - rest\ mass\ energy$$

2. Potential Energy

$$gravity = -\frac{G\,M\,m}{r} \qquad or \qquad mgh$$

$$electric = \frac{k\,Q\,q}{r} \qquad or \qquad qV$$

$$elastic = \frac{1}{2} k\ x^2$$

$$\ldots$$

3. Internal Energy

$$thermal = mass\,(thermal\ capacity)\,temperature$$

$$thermal = \frac{3}{2} N\ k_{Boltzman} T \quad for\ a\ monatomic\ ideal\ gas$$

$$chemical = heat\ energy\ released\ when\ a\ fuel\ is\ burned$$

$$nuclear = m\,c^2$$

$$\ldots$$

Sometimes a little of the energy that we start with disappears into internal energy and we speak of energy “losses” in the sense that the missing energy is no longer readily available to do useful work. For example the transformation of external work into thermal energy by motion-impeding forces which causes increased agitation of atomic motion (increased kinetic energy); or the permanent deformation of objects during collisions which causes the reshaping of chemical molecular bonds (increased potential energy).

Energy and time

The reader will have noticed that nowhere in the conservation of energy does the time interval duration of the process appear. That is simply due to the system's energy stability over time; without some energy-exchange process occurring with the rest of the universe a system's energy level stays fixed and constant, there is no means for it to change.

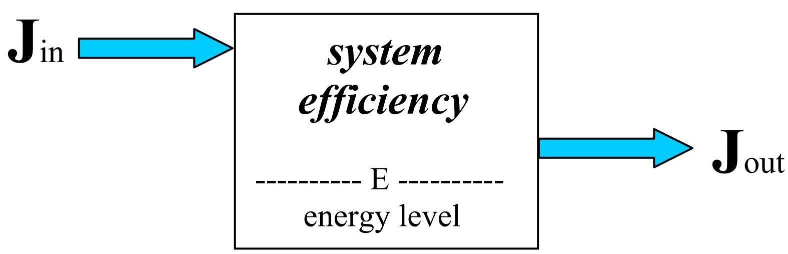
So if we wish to know the time duration of a process, we must track the ***rate*** at which the energy change of the system occurs. This rate is called ***power*** and is defined $P \equiv \Delta E/\Delta t$. The unit of power is Joule per second which is called a Watt, abbreviated W.

example: Suppose a system changes its energy by a total of 24 J and does so at the ***rate*** of 3 Joule every second or Watt. How long does the ΔE process take?
solution: From the definition of power $\Delta t = \Delta E/P$ = (24 Joule) / (3 Watt) = 8 s.

Power in terms of the work force involved is, $Power \equiv \dfrac{\Delta E}{\Delta t} = \dfrac{\vec{F} \bullet \vec{\Delta x}}{\Delta t} = \vec{F} \bullet \overline{velocity}$

Energy Efficiency

The efficiency of any system, normally a machine, is defined as the ***ratio*** of the total energy (Joules) out of a system to the total energy (Joules) put into the system, while the system remains at equilibrium. It is a pure ratio and therefore has no units.



$$efficiency \equiv \frac{J_{out}}{J_{in}} = \frac{\dfrac{J_{out}}{\Delta t}}{\dfrac{J_{in}}{\Delta t}} = \frac{Power_{out}}{Power_{in}}$$

Note: Efficiency also keeps account of the 'loss' factor which is 1 - efficiency.

example:
Consider a motor requiring 1800 W input while its output shaft delivers 2.0 horsepower.

$$motor's\ \ efficiency = \frac{Power_{out}}{Power_{in}} = \frac{2.0\ hp\ x\ 745.7\ Watt/\ hp}{1800\ Watt} = 0.83\ \ or\ \ 83\,\%$$

The motor's 'loss' factor is 1 - 0.83 = 0.17 or 17 % which means that the system itself dissipates 17% of its input or 0.17 x 1800 W which is 309 Watt or 309 Joule every second.

Energy Efficiency and Related Ratios

1. Mechanical engineers like to refer to mechanical force advantage.

$$efficiency \equiv \frac{J_{out}}{J_{in}} = \frac{Force_{out} \cdot displacement_{out}}{Force_{in} \cdot displacement_{in}} = \frac{\dfrac{Force_{out}}{Force_{in}}}{\dfrac{displacement_{in}}{displacement_{out}}} = \frac{AMA}{IMA}$$

where forces and their corresponding displacements are taken to be parallel to each other. The force ratio is called the **A**ctual **M**echanical **A**dvantage. The displacement ratio is called the **I**deal **M**echanical **A**dvantage because it is the maximum possible **M**echanical **A**dvantage which occurs when there is zero energy dissipation (zero transformed to IE) in the system, that is zero energy is absorbed by the system, that is when the efficiency is 1.

A certain device has efficiency 0.80. The ***input*** force acts over a displacement of 10 cm while the ***output*** force acts over a distance of 2 cm. What is the AMA of this device ?

solution:

$$IMA \equiv \frac{displacement_{in}}{displacement_{out}} = \frac{10\,cm}{2\,cm} = 5$$

$$AMA \equiv \frac{Force_{out}}{Force_{in}} = efficiency \cdot IMA = (0.80) \cdot 5 = 4$$

2. Electrical engineers like to refer to the gain of a system or amplifier stage.
Note that electric power is current times potential difference or 'volts' that is P = IV.

$$ratio\;\; or\;\; gain \equiv \frac{Power_{out}}{Power_{in}} = \frac{I_{out} \cdot V_{out}}{I_{in} \cdot V_{in}}$$

$$or \qquad Power_{gain} = Current_{gain} \cdot Voltage_{gain}$$

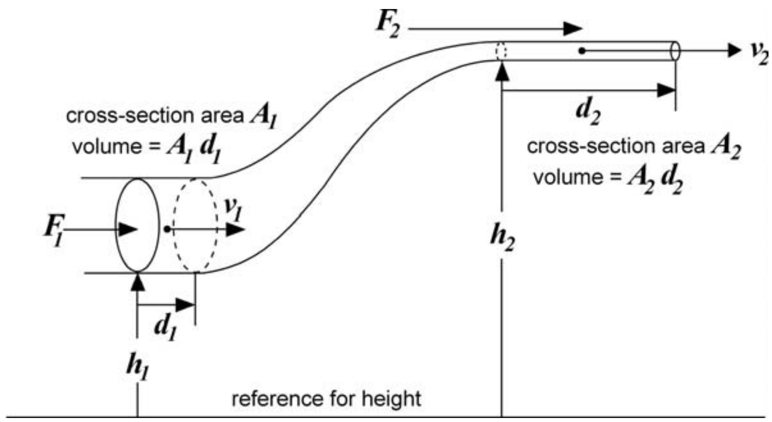
A certain amplifier stage has ***input*** current 14 ampere while its voltage gain is 7 and its overall power gain is 4. What is the ***output*** current of this amplifier ?
solution:

$$Power_{gain} = I_{gain} \cdot V_{gain} \qquad 4 = \frac{I_{out}}{14\;ampere} \cdot 7 \qquad I_{out} = 8\;ampere.$$

Note that in order to obtain an energy or power ***gain*** from a system, there must of course be an energy source inside the system (for example a battery) as well as an energy sink.

Fluid Flow Energy

Consider a parcel of fluid flowing along an (imaginary) pipe as shown in the diagram.



Select the parcel of fluid to be the system of interest, all else is external to the system. Let an external force F_1 push on the fluid from the left and the fluid push with a force F_2 on the right. The work done ***on*** the system by the external force F_1 acting through a distance d_1 is F_1d_1.
The work done ***by*** the system on the external by the force F_2 through a distance d_2 is F_2d_2.
The ***net*** work done on the parcel of fluid is thus $F_1d_1 – F_2d_2$ which is energy into the system.

We write the conservation of energy: (frictional forces may generate thermal energy ⟶ heat)

$$J_{in\ or\ out} = \Delta E_{kinetic} + \Delta E_{potential} + \Delta E_{\text{internal}}$$

$$F_1 d_1 - F_2 d_2 = \left(\frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2\right) + \left(m g h_2 - m g h_1\right) + \left(thermal\ energy\right)$$

We now restrict the analysis to an ***incompressible***, viscous fluid; its density (ρ ≡ mass/volume) is a constant everywhere in the fluid. This means as the fluid moves along the pipe through a distance d_1 on the left and simultaneously through a distance d_2 on the right, the amount of mass $\rho A_1 d_1$ around point 1 in the fluid is equal to the mass $\rho A_2 d_2$ around point 2. It is as if the parcel of mass $\rho A_1 d_1 = \rho A_2 d_2$ moved from point 1 to point 2 due to the energy put into the system. We divide the above energy equation on both sides by the volume $A_1 d_1 = A_2 d_2$ of this mass parcel.

$$\frac{F_1 d_1}{A_1 d_1} - \frac{F_2 d_2}{A_2 d_2} = \left(\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2\right) + \left(\rho g h_2 - \rho g h_1\right) + \left(thermal\ energy\right) \qquad note: m = \rho \cdot volume$$

$$note \quad \frac{F_1}{A_1} = P_1 \ \ the\ pressure\ in\ the\ fluid\ at\ \text{point 1}$$

$$\frac{F_2}{A_2} = P_2 \ \ the\ fluid\ pressure\ at\ \text{point 2}$$

$$P_1 - P_2 = \left(\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2\right) + \left(\rho g h_2 - \rho g h_1\right) + \left(thermal\ energy\right)$$

and rearrange to obtain Bernoulli's equation for fluid flow:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 + \begin{pmatrix} thermal\ energy \\ per\ unit\ volume \end{pmatrix} \qquad units: energy\ density\ \frac{J}{m^3} = \frac{N}{m^2}$$

In any closed system, energy is neither created nor destroyed, it can only transform.
An ***ideal fluid*** has both constant density and zero internal friction (no thermal energy losses).

Application: Pressure and Power for Fluid Flow

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 + \left(\begin{matrix} thermal\ energy \\ per\ unit\ volume \end{matrix}\right) \qquad units: pressure\ \frac{N}{m^2} = \frac{J}{m^3}$$

In the human body's cardiovascular system, as blood flows farther from the heart (pump source point 1) more thermal energy leaves the system in the form of body heat so the pressure and the average speed of the blood flow decrease. However, for points sufficiently close together energy losses due to heat may be negligible thereby greatly simplifying predictions about fluid flow.

Conservation of mass and flow rate: Rate of mass into a vessel = Rate of mass out of a vessel.

$$mass\ flow\ rate \equiv \frac{mass}{\Delta t} = \frac{\rho \cdot volume}{\Delta t} \qquad units\ \frac{kg}{s}$$

$$volume\ flow\ rate \equiv \frac{volume}{\Delta t} = \frac{Area \cdot \Delta x}{\Delta t} = A \cdot v \qquad units\ \frac{m^3}{s}$$

Fluid's resistance to flow and Poiseuille's Law:

$$volume\ flow\ rate = \frac{pressure\ drop\ across\ vessel}{\text{resistance}\ of\ the\ fluid\ to\ flow}; \quad Q = \frac{\Delta P}{R} \qquad for\ \text{laminar}\ flow: \ R = \frac{8\eta L}{\pi r^4}$$

where: η = measure of fluid's internal friction; L=Length of the vessel; r = radius of the vessel.

¿ Estimate the pressure difference when human blood flows where the diameter of the vessel carrying the fluid (for example the aorta with diameter ≈ 12 mm) decreases by a factor of 5 %? The human heart pumps blood at a volume flow rate of about 5.0 liter/min or 83 cm^3/s.
Blood: density = 1.05 gram/cm^3 = 1050 kg/m^3 ; coefficient of viscosity $\eta = 3.5 \times 10^{-3}$ Pa·s.

A. between two regions, with no internal friction (ideal fluid) ; (both regions at same height $h_1 = h_2$)

$$v_1 = \frac{volume\ flow\ rate}{cross\ \text{section}\ area} = \frac{83\ cm^3/\text{sec}}{\pi (0.6\, cm)^2} = 73\ \frac{cm}{s} \qquad and \qquad v_2 = 81\ \frac{cm}{s}$$

$$P_1 - P_2 = \frac{1}{2}\rho v_1^2 - \frac{1}{2}\rho v_2^2 = \frac{1}{2}\left(1050\frac{kg}{m^3}\right)\left\{\left(0.813\frac{m}{s}\right)^2 - \left(0.734\frac{m}{s}\right)^2\right\} = 64\ Pa$$

B. for a given region, with internal friction (streamline: laminar flow with negligible turbulence):

Resistance of a 5-cm long vessel at radius 6.0 mm $R_1 = \dfrac{8\,(3.5\,x\,10^{-3}\ Pa \cdot s)(5\,x\,10^{-2}\ m)}{\pi\,(6.0\,x\,10^{-3}\ m)^4} = 3.44\,x\,10^5\ \dfrac{Pa \cdot s}{m^3}$

Pressure difference at radius 6.0 mm, $\Delta P_1 = Q \cdot R = 83 \times 10^{-6}\ m^3/s \cdot 3.44 \times 10^5 Pa \cdot s/m^3 = 29$ Pa

Resistance of a 5-cm long vessel at radius 5.7 mm $R_2 = \dfrac{8\,(3.5\,x\,10^{-3}\ Pa \cdot s)(5\,x\,10^{-2}\ m)}{\pi\,(5.7\,x\,10^{-3}\ m)^4} = 4.22\,x\,10^5\ \dfrac{Pa \cdot s}{m^3}$

Pressure difference at radius 5.7 mm, $\Delta P_2 = Q \cdot R = 83 \times 10^{-6}\ m^3/s \cdot 4.22 \times 10^5 Pa \cdot s/m^3 = 35$ Pa

C When the pressure in the blood vessels rises, the heart has to pump more to keep the blood circulating. ¿ Estimate the power increase (%) the heart must deliver to maintain the same flow rate to the organs?

Note: pressure increase of 35.1/28.7 = 1.23 (23 %); velocity increase of 81.3/73.4 = 1.11 (11 %).

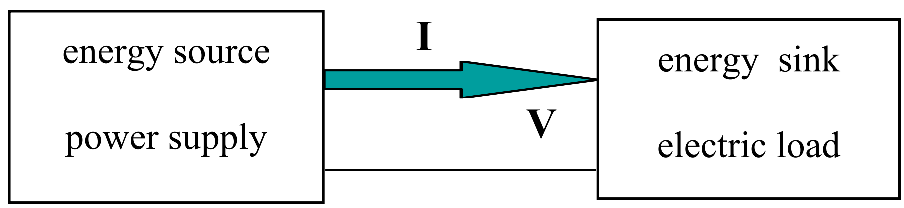
$$Answer: \frac{Power'}{Power} = \frac{F'}{F}\frac{v'}{v} = \frac{P'A'v'}{P A v} = \frac{P'Q'}{PQ} = 1.23 * 1 = 1.23 = \left(\frac{1}{0.95}\right)^4 \quad or \quad 23\,\% \quad more\ power.$$

A small amount (5%) of arterial occlusion can have a surprisingly large effect on the heart pump!

Application: Electric Energy & Electricity Laws

Kirchhoff's Laws concerning the currents and potential differences in electric circuits can be derived from the principle of conservation of energy. From the definitions of current and potential difference, it follows that electric power is current times voltage, P = I V.

Consider an energy source or power supply connected to an energy sink or electric load. Let **I** be the current that flows between the source and sink and **V** the potential difference (volt) that is common to both the source terminals and the sink terminals, as shown.



We write the conservation of energy as: $\frac{\Delta E_{source}}{\Delta t} = \frac{\Delta E_{sink}}{\Delta t}$ which are powers.

The power ***out*** of the source which is transferred ***into*** the sink is P_{total} = **I V**.

Thus the power supply exchanges $\mathbf{J}_{\text{in or out}} = \mathbf{IV}\cdot\Delta t$ energy with the sink part of system. Now, when the electric load consists of several separate loads, the ***total*** power dissipated in the sink will be $I_1V_1 + I_2V_2 + I_3V_3 + \ldots$ as the energy must go somewhere in the sink. The above conservation of energy statement then becomes: $\mathbf{IV} = I_1V_1 + I_2V_2 + I_3V_3 + \ldots$

<u>Case 1</u> - Series Circuit

The electrical load consists of several loads all of which take the same common current **I**. Then the total power involved is $\mathbf{IV} = I_1V_1 + I_2V_2 + I_3V_3 + \ldots$ where all currents are **I**. Since the common current cancels, we obtain $\mathbf{V} = V_1 + V_2 + V_3 + \ldots$ which states that around any closed loop the sum of the sink (dissipating) voltages is equal to the sum of the source (supply) voltage(s); this is known as Kirchhoff's **Voltage Law** (KVL) or the Circuit Loop Law, or the sum of the voltage drops is equal to the sum of the voltage rises. It can also be stated as: the sum of the voltages around a complete and closed loop is zero; where supply voltages have one sign (±) and sink voltages have the opposite sign.

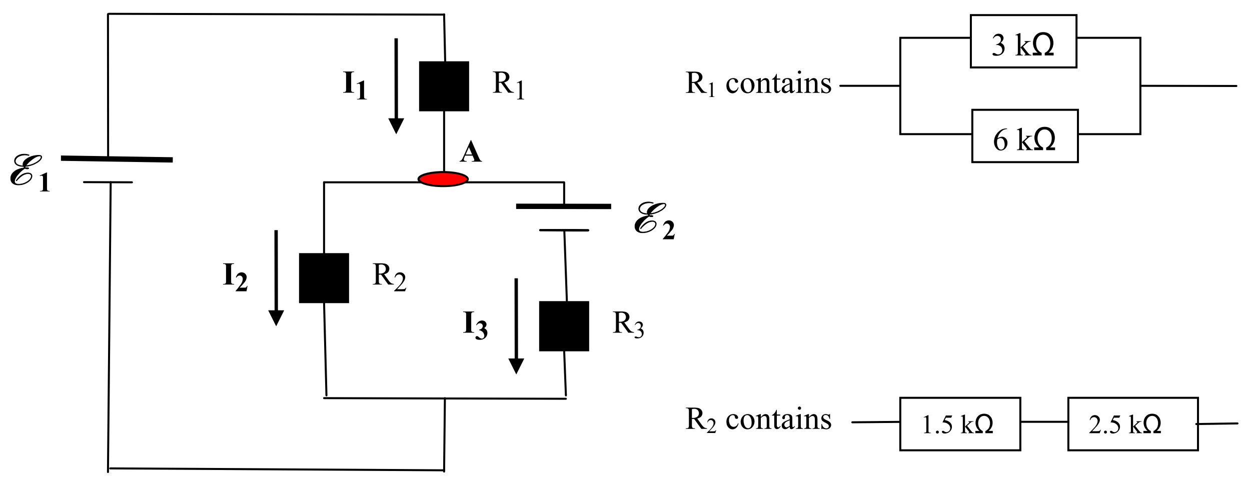
<u>Case 2</u> - Parallel Circuit

The electrical load consists of several loads all of which take the same common voltage **V**. Then the total power involved is $\mathbf{IV} = I_1V_1 + I_2V_2 + I_3V_3 + \ldots$ where all voltages are **V**. Since the common voltage cancels, we obtain $\mathbf{I} = I_1 + I_2 + I_3 + \ldots$ which states that at any junction the sum of the outgoing currents is equal to the sum of the incoming current(s); this is known as Kirchhoff's **Current Law** (KCL) or the Circuit Junction or Node Law. It can also be stated as: the sum of the algebraic currents at a junction or node is zero; where incoming currents have one sign (±) and outgoing currents have the opposite sign.

Example: Kirchhoff's Voltage and Current Law.

Kirchhoff's Voltage Law: around any closed electrical circuit loop,
KVL: the sum of the voltage drops is equal to the sum of the voltage rises.
Kirchhoff's Current Law: at any junction node in an electrical circuit,
KCL: the sum of the incoming currents equals the sum of the outgoing currents.

Apply these two circuit laws to the circuit shown and demonstrate conservation of energy.



Examine the 3 currents at node **A** and apply Kirchhoff's Current Law as stated above.
Similarly examine the two loops and apply Kirchhoff's Voltage Law as stated above.
Apply Ohm's rule (V=IR or I=GV) for the voltage drops across the energy sinks.

KCL at node A :	$I_1 = I_2 + I_3$
KVL for loop with $\mathscr{E}_1$:	$\mathscr{E}_1 = I_1 R_1 + I_2 R_2$
KVL for loop with $\mathscr{E}_2$:	$\mathscr{E}_2 + I_3 R_3 = I_2 R_2$

Given: $\mathscr{E}_1$ **= 11 volt** $\mathscr{E}_2$ **= 33 volt** G_3 **= 1/6 mS**

Note: R_1 **= 2 kΩ** R_2 **= 4 kΩ** R_3 **= 6 kΩ**

With these given values, there results 3 equations in the 3 unknowns.
The reader should solve the mathematics for the three unknown currents:

Answer: I_1 **= −0.5 mA** I_2 **= +3.0 mA** I_3 **= −3.5 mA**

Having solved for the three unknown currents, we know everything that can be known about each individual device in this electrical circuit.

We fill in a table with the values pertaining to each device in the circuit:

Device ID	potential difference V volt	current I mA	resistance R=V/I kΩ	conductance G = I/V mS	power IV mW
R_1	−1	−0.5	**2**	1/2	0.5
R_2	12	3.0	**4**	1/4	36
R_3	−21	−3.5	6	**1/6**	73.5
$\mathscr{E}_1$	**11**	−0.5			−5.5
$\mathscr{E}_2$	**33**	3.5			115.5

Bolded values are given or pre-determined values, all other values are derived with the 3 currents.

The total power consumed by the energy sinks is $P_{total} = 0.5 + 36 + 73.5 = 110$ mW.

The total power supplied by the energy sources is $P_{total} = 115.5 + (-5.5) = 110$ mW.

Thus the same net amount of energy per second is leaving the supply source(s) as is being consumed or entering the sinks, and we see that energy is indeed conserved.

Note that I_1 turned out to be a negative number, which really means the actual current is in the opposite direction of the arrow of I_1, (arrow direction is positive). The current in the loop containing the $\mathscr{E}_1$ energy source is running counter-clockwise!

That current is ***entering*** the $\mathscr{E}_1$ energy source which means $\mathscr{E}_1$ is being charged using the energy from the second $\mathscr{E}_2$ energy source or power supply. $\mathscr{E}_2$ is being depleted at the rate of 115.5 milliJoules per second of which 110 are absorbed by the three energy sinks and 5.5 mW are being pumped into the first $\mathscr{E}_1$ power supply.

Application: Electric Power Factor

For time-varying signals, circuit voltages/currents may not rise or fall together in step.

We start with conservation of energy: $\mathrm{P_{source} = P_{Load}}$

and use $\boldsymbol{E}$ for the applied emf,

small letters for time-dependent quantities,

capital letters for quantities independent of time

and subscript $\mathrm{I_0}$ to denote amplitude values.

$$ie = iv_{total}$$

$$\textit{for a series RCL circuit} \quad ie = iv_R + iv_L + iv_C$$

$$\textit{let the phase angle } \phi = \textit{angle between the applied emf } (e) \textit{ and the resulting current } \{\ (i) = I_0 \sin(\omega t)\ \}.$$

$$I_0 \sin(\omega t) \bullet E_0 \sin(\omega t + \phi) = I_0 \sin(\omega t) \bullet V_{R0} \sin(\omega t) + I_0 \sin(\omega t) \bullet V_{L0} \sin(\omega t + 90^\circ) + I_0 \sin(\omega t) \bullet V_{C0} \sin(\omega t - 90^\circ)$$

$$I_0 \sin(\omega t) \bullet E_0 \sin(\omega t + \phi) = I_0 \sin(\omega t) \bullet V_{R0} \sin(\omega t) + I_0 \sin(\omega t) \bullet V_{L0} \cos(\omega t) \qquad - I_0 \sin(\omega t) \bullet V_{C0} \cos(\omega t)$$

$$I_0 \sin(\omega t) \bullet E_0 \sin(\omega t + \phi) = I_0 \sin(\omega t) \bullet V_{R0} \sin(\omega t) + I_0 \sin(\omega t) \bullet (V_{L0} - V_{C0}) \bullet \cos(\omega t) \qquad \textit{equation } (1)$$

Expand, using the trigonometric identity $\sin(\omega t + \phi) \equiv \sin(\omega t) \bullet \cos(\phi) + \cos(\omega t) \bullet \sin(\phi)$

We need time averaging <rms> over cycles:

$$< \sin(\omega t) > = 0$$

$$< \cos(\omega t) > = 0$$

$$< \sin^2(\omega t) > = 1/2$$

$$< \cos^2(\omega t) > = 1/2$$

$$< \sin(\omega t) \bullet \cos(\omega t) > = 0 \quad \textit{all cross terms}$$

(A) Apply <rms> averaging to the power equation (1).

$$I_0 E_0 \frac{1}{2} \cos\phi = I_0 V_{R0} \frac{1}{2} + \textit{zero}$$

$$\frac{I_0}{\sqrt{2}} \frac{E_0}{\sqrt{2}} \cos\phi = \frac{I_0}{\sqrt{2}} \frac{V_{R0}}{\sqrt{2}} \qquad \textit{or} \qquad \cos\phi = \frac{V_{R\,rms}}{E_{rms}}$$

Power dissipation in circuit: $P_{average} = I_{rms} E_{rms} \cos\phi = I_{rms}^2 R.$

The total power from the power supply is dissipated in the resistive component of the circuit. Although energy is supplied to a capacitor or an inductor during one part of the cycle, this energy is returned during another part of the cycle and the average power input is zero to reactive components. It is similar to the energy sloshing back-and-forth between kinetic and potential energy in an oscillator.

(B) Square the power equation (1) and apply <rms> averaging.

$$I_0^2 \sin^2 \omega t \cdot E_0^2 \sin^2(\omega t + \phi) = I_0^2 \sin^2 \omega t \cdot V_{R0}^2 \sin^2 \omega t + I_0^2 \sin^2 \omega t \cdot (V_{L0} - V_{C0})^2 \cdot \sin^2 \omega t + cross\ terms$$

apply <rms> averaging:

$$I_0^2 \frac{1}{2} \bullet E_0^2 \frac{1}{2} = I_0^2 \frac{1}{2} \bullet V_{R0}^2 \frac{1}{2} + I_0^2 \frac{1}{2} \bullet (V_{L0} - V_{C0})^2 \frac{1}{2} + zero$$

$$I_{rms}^2 E_{rms}^2 = I_{rms}^2 V_{Rrms}^2 + I_{rms}^2 (V_{Lrms} - V_{Crms})^2$$

$$1 = \left\{ \frac{V_{Rrms}}{E_{rms}} \right\}^2 + \left\{ \frac{V_{Lrms} - V_{Crms}}{E_{rms}} \right\}^2$$

$$1 = \cos^2 \phi \quad + \quad \sin^2 \phi \qquad\qquad from\,(\mathrm{A}):\ \cos\phi = \frac{V_{Rrms}}{E_{rms}}$$

The term $\cos\phi$ is called the ***power factor*** of the circuit element.

element	symbol	ϕ	$\cos\phi$	Average Power
resistance	R	0°	1	$I_{R\,rms} V_{R\,rms} = I_{R\,rms}^2 R$
capacitance	C	-90°	0	0
Inductance	L	+90°	0	0

(C) Note on <rms> values:

Ammeters and voltmeters are in practice usually calibrated to read <rms> or effective values directly. The total impedance of the load is the ratio of the applied emf to the current:

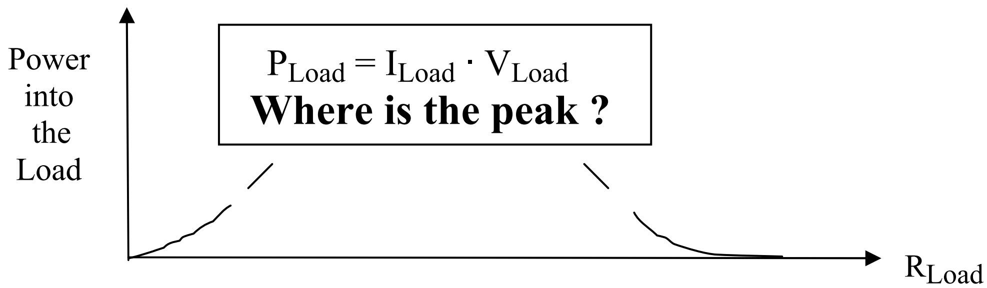
$$Z \equiv \frac{E_{rms}}{I_{rms}} = \frac{E_0}{I_0}$$

***E* = IZ** with $Z^2 = R^2 + (X_L - X_C)^2$ and $V_R = IR$; $V_L = IX_L$; $V_C = IX_C$.

Power dissipation in circuit: $P_{average} = I_{rms} E_{rms} \cos\phi = I_{rms}^2 Z \cos\phi = I_{rms}^2 R$.

Application: Power Transfer - Impedance Matching

Consider a resistive load R_{Load} attached to a power supply *Emf* which has internal resistance R_{supply}. Investigate the amount of power transferred ***from*** the power supply ***into*** the load by plotting the power absorbed into the load versus the load's resistance.



small R_{Load}	large R_{Load}
$0 \Leftarrow R_{Load}$	$R_{Load} \Rightarrow \infty$
$V_{Load} = 0$	$I_{Load} = 0$
$P_{Load} = 0$	$P_{Load} = 0$

$$P_{Load} = I_{Load}^2 \, R_{Load} \qquad \text{with KVL:} \quad P_{Load} = \left(\frac{E}{R_{Supply} + R_{Load}} \right)^2 R_{Load}$$

How does the P_{Load} vary with the resistive size of the load itself, that is with R_{Load} **?**

Let $X \equiv R_{Load}$ and $R_s \equiv R_{Supply}$ then the power absorbed: $P_{Load} = E^2 X \left(R_s + X\right)^{-2}$

$$\frac{dP_{Load}}{dX} = E^2 \left[X\left\{ -2\left(R_s + X\right)^{-3} \right\} + \left\{ R_s + X \right\}^{-2} (1) \right]$$ Find where the slope is zero.

$$\frac{dP_{Load}}{dX} = E^2 \, \frac{\left(R_s + X\right) - 2X}{\left(R_s + X\right)^3} = E^2 \, \frac{R_s - X}{\left(R_s + X\right)^3} \mathbf{= 0 \quad when \quad X = R_{Supply}}$$

$R_{Load} = R_{Supply}$ is called ***Impedance Matching*** for Maximum Power Transfer, at which the power into the load is $P_{Load} = \{ E/(2R_{Supply}) \}^2 R_{Load} = ½I_{sc}½V_{oc}$.

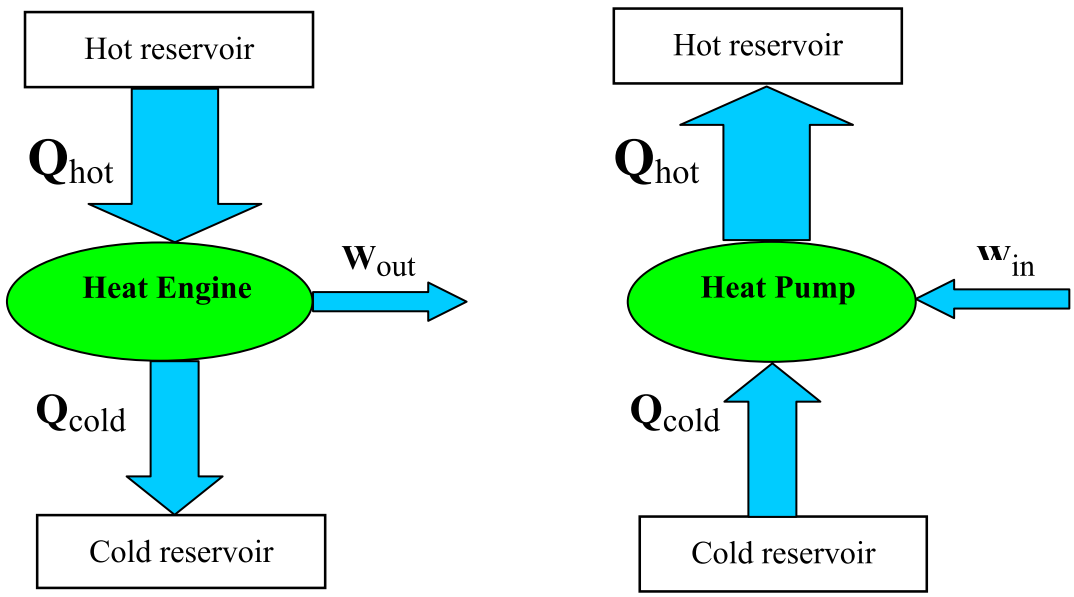
Problem: What maximum power can be transferred out of a power supply with **e**lectro-**m**otive-**f**orce *Emf* =12 volt and internal resistance 18 ohm **?**

Solution: $P_{max} = ½ \{ I_{short\text{-}circuit} = 12 \text{ volt}/18 \} ½ \{ V_{open\text{-}circuit} = 12 \text{ volt} \}$
$P_{max} = ½ \{ 0.666 \text{ A} \} ½ \{ 12 \text{ volt} \} = 2 \text{ Watt}$

Application: Thermal Energy Devices

Devices which depend for their successful operation on a temperature difference can be readily understood using the principle of conservation of energy. Such thermal devices can be classified into two general categories: heat engines and heat pumps.

Suppose one has a high-temperature reservoir of thermal energy and also a low-temperature reservoir, between which a device operates by virtue of the temperature difference between the reservoirs. The device must be thermally connected to both reservoirs (T_{hot} and T_{cold}) so that thermal energy (**Q**) flows through the device, as shown in the diagrams.



Conservation of energy: $\mathbf{Q}_{hot} = \mathbf{Q}_{cold} + W_{out}$ $\qquad$ $\mathbf{Q}_{hot} = \mathbf{Q}_{cold} + W_{in}$

$$efficiency_{heat\ engine} \equiv \frac{W_{out}}{Q_{hot}} = \frac{Q_{hot} - Q_{cold}}{Q_{hot}} \qquad COP_{refrigerator} \equiv \frac{Q_{cold}}{W_{in}} = \frac{Q_{cold}}{Q_{hot} - Q_{cold}}$$

COP means Coefficient of Performance

$$COP_{heat\ pump} \equiv \frac{Q_{hot}}{W_{in}} = \frac{Q_{hot}}{Q_{hot} - Q_{cold}}$$

Kelvin's absolute temperature means Q is proportional to T, which permits all three ratios to be written in terms of the temperatures of the reservoirs; the ***maximum*** ratios are:

$$efficiency_{heat\ engine} = \frac{\Delta T}{T_{hot}} \qquad COP_{refrigerator} = \frac{T_{cold}}{\Delta T} \qquad COP_{heat\ pump} = \frac{T_{hot}}{\Delta T}$$

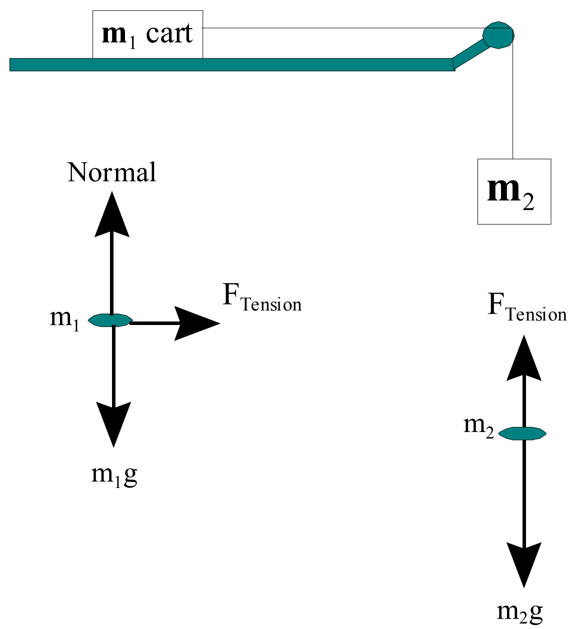
Note, these results say nothing about: the working substance, the material the heat device is made of, or how a device is constructed; it is only the absolute temperatures that matter. Carnot's Theorem (1824) says that no practical thermal device can exceed these ratios.

Application: Rotational Energy and Inertia

Consider a cart on four wheels (mass m_1) on a level track connected by a light string over a pulley to a second suspended mass m_2 (see diagram below). Assume both cart and pulley are frictionless, and ignore the mass of the string. Assume the string is non-stretchable, then both masses have the same acceleration. The tension force $F_{tension}$ in the string is the same on both sides of the pulley, because we are neglecting the friction in the pulley for the moment. Note: the suspended weight m_2g is the ***applied force*** causing the entire system to accelerate. Analyze the motion of this system from the point of view using: (a) forces and (b) energy.

A. **Force** analysis of the motion of the system (cart plus suspended mass).

The free-body diagrams for the cart and suspended mass are illustrated to assist in analysis.



Cart	**Suspended Mass**
$F_{net} = F_{tension}$	$F_{net} = m_2g - F_{tension}$
$m_1a_1 = F_{tension}$	$m_2a_2 = m_2g - F_{tension}$

Combining these two equations by substituting $F_{tension}$ and accounting for the friction:

$$m_2 g - F_{friction} = (m_1 + m_2)\, a \qquad \textbf{(1)}$$

Note, Equation (1) states: the **net** force (applied force m_2g less the opposing friction) and the acceleration ***a*** of the system are proportional with combined mass constant of $(m_1 + m_2)$. We may view the combined cart-plus-suspended mass as one system of total mass $m_1 + m_2$. In this analysis, the cause of any rotation of the pulley and the cart wheels has been ignored.

B. Energy analysis of the motion of the system (cart plus suspended mass).

As the system moves it gains kinetic energy of motion at the expense of the potential energy of position of the suspended weight m_2g. When the suspended weight falls through vertical distance *y* its potential energy decreases by m_2gy, and using the **conservation of energy**:

$$m_2 g\, y = \quad KE^{cart}_{translation} \quad + \; KE^{pulley}_{rotation} \; + \; KE^{wheels}_{rotation} + \; IE_{friction}$$

$$m_2 g\, y = \frac{1}{2}\left(m_1 + m_2\right) v^2 + \frac{1}{2} I_{pulley}\omega^2 + \frac{1}{2} I_{wheels}\omega^2 + RF_{friction}\theta$$

R = radius of pulley or wheels.
For non-slip rotational motion: $\omega = \frac{v}{R} \quad and\ thus \quad I\omega^2 = \frac{I}{R^2} v^2$

$$m_2 g\, y = \frac{1}{2}\left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{pulley} + 4\left(\frac{I}{R^2}\right)_{wheel} \right] v^2 + RF_{friction}\theta$$

We differentiate this energy equation with respect to time and recognize that:

$$\dot{y} = \frac{dy}{dt} = system\ velocity = v$$

$$\dot{v} = \frac{dv}{dt} = system\ acceleration = a$$

$$\dot{\theta} = \frac{d\theta}{dt} = rotational\ velocity = \omega = \frac{v}{R}$$

$$m_2 g \dot{y} = \frac{1}{2}\left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{pulley} + 4\left(\frac{I}{R^2}\right)_{wheel} \right] 2\, v\, \dot{v} + RF_{friction}\dot{\theta}$$

$$m_2 g \;=\; \left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{pulley} + 4\left(\frac{I}{R^2}\right)_{wheel} \right] a \;+\; F_{friction} \qquad \textbf{(2)}$$

Note: Equation (**2**) is the same as (**1**) stating force is proportional to acceleration but with a different mass constant which incorporates the ***rotational*** inertia of the pulley and wheels.

Application: Oscillator Energy

Harmonic Motion with laminar damping.
Damped, driven harmonic oscillator.

Consider a damped oscillator forced to operate by a sinusoidal driver force $F_o \sin(\omega t)$
Analyze the motion of this system from the point of view using: (a) forces and (b) energy.

A. **Force** analysis of the motion of the system (driver force & damped oscillator).

B. **Energy** analysis of a driven damped oscillator system (with energy source).

C. Show that the **quality** of an oscillator $Q = \dfrac{\textit{frequency at resonance}}{\text{full width at half maximum power}}$.

A. **Force** analysis of the motion of the system (force driving a damped oscillator).

Show **x = Asin(ωt+ϕ)** satisfies the equation of motion of a ***driven*** harmonic oscillator.

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F_o \sin\omega t$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_o \sin\omega t}{m}$$

$$\frac{c}{m} \equiv \frac{1}{\tau} \qquad \frac{k}{m} \equiv \omega_o^2$$

Substituting the first and second differentiation of the proposed solution x = Asin(ωt+ϕ):

$$\left(\omega_o^2 - \omega^2\right) A\sin\left(\omega t + \varphi\right) + \frac{\omega}{\tau} A\cos\left(\omega t + \varphi\right) = \frac{F_o}{m}\sin\left(\omega t\right)$$

using: sin(ωt+ϕ) = sinωt cosϕ + cosωt sinϕ
cos(ωt+ϕ) = cosωt cosϕ - sinωt sinϕ

$$\left[\left(\omega_o^2 - \omega^2\right)\cos\varphi - \frac{\omega}{\tau}\sin\varphi\right] A\sin\omega t + \left[\left(\omega_o^2 - \omega^2\right)\sin\varphi + \frac{\omega}{\tau}\cos\varphi\right] A\cos\omega t = \frac{F_o}{m}\sin\omega t$$

This equation can only be satisfied for all time t if the coefficients of sinωt and cosωt are each zero, which establishes A and the phase angle ϕ:

$$A = \frac{F_o / m}{\left(\omega_o^2 - \omega^2\right)\cos\varphi - \frac{\omega}{\tau}\sin\varphi} = \frac{F_o / m}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega}{\tau}\right)^2}}$$

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = -\frac{\omega / \tau}{\omega_o^2 - \omega^2}$$

where we used $\sin^2\phi + \cos^2\phi \equiv 1$ in order to obtain:

$$\sin\varphi = \frac{-\omega/\tau}{\sqrt{\left(\omega_o^2-\omega^2\right)^2+\left(\frac{\omega}{\tau}\right)^2}} \quad \cos\varphi = \frac{\omega_o^2-\omega^2}{\sqrt{\left(\omega_o^2-\omega^2\right)^2+\left(\frac{\omega}{\tau}\right)^2}}$$

Thus under the influence of an external driving force $\mathbf{F = F_o \sin \omega t}$ the system response is:

Position: $\mathbf{x = A \sin(\omega t + \phi)}$ displacement always lags ($\phi < 0$) the driving force

$$Amplitude\ A = \frac{F_o/m}{\sqrt{\left(\omega_o^{\,2}-\omega^2\right)^2+\left(\frac{\omega}{\tau}\right)^2}} \qquad \tan\varphi = -\frac{\omega/\tau}{\omega_o^{\,2}-\omega^2}$$

a) At low driving frequency $\omega << \omega_o$
Response controlled by spring (k)

$$A = \frac{F_o/m}{\omega_0^{\,2}} = \frac{F_o}{k} \qquad \varphi \to 0$$

b) At resonance driving frequency $\omega = \omega_o$
Response controlled by damping (c)

$$A = \frac{F_o/m}{\omega_o/\tau} = \frac{F_o/\omega_o}{c} \qquad \varphi \to -\frac{\pi}{2}$$

c) At high driving frequency $\omega >> \omega_o$
Response controlled by inertial mass (m)

$$A = \frac{F_o/m}{\omega^2} = \frac{F_o/\omega^2}{m} \qquad \varphi \to -\pi$$

Note in particular the ratio of the response at resonance to the response at zero frequency. It turns out to be the quality of the oscillator at the resonance frequency:

$$\frac{response\ at\ resonance}{response\ at\ zero\ frequency} = \frac{A(\omega=\omega_o)}{A(\omega=0)} = \frac{1/\omega_o^2}{\omega_o/\tau} = \omega_o\tau = Q_{resonance}$$

B. **Energy** analysis of a driven damped oscillator system (i.e. with energy source).

Derive the power absorption $P(\omega)$ for a driven damped harmonic oscillator.

For the motion of a forced or driven oscillator we have:

force $\mathbf{F = F_o \sin \omega t}$; position $\mathbf{x = A \sin(\omega t + \phi)}$; velocity $\mathbf{v = \omega A \cos(\omega t + \phi)}$

Note the phase angle ϕ of the position $\boldsymbol{x}$ relative to the driving force $\boldsymbol{F}$ and an additional 90° for the velocity $\boldsymbol{v}$ which is the derivative of the position.

The time averaged power is the time average of the force times the velocity

$\text{Power} = \langle F_{\text{external}} \cdot \text{velocity} \rangle$ $\langle\rangle \equiv$ **time average**

$\text{Power} = \langle (F_o \sin\omega t) \cdot \{ \omega A \cos(\omega t + \phi) \} \rangle$

$\cos(\omega t + \phi) = \cos\omega t \cos\phi - \sin\omega t \sin\phi$

$\text{Power} = F_o\, \omega\, A \langle \sin\omega t \cos\omega t \cos\phi - \sin\omega t \sin\omega t \sin\phi \rangle$

The time average of $\sin\omega t \cos\omega t$ is zero, that is $\langle \sin\omega t \cos\omega t \rangle = 0$
and the time average of $\sin^2\omega t$ is one half, that is $\langle \sin^2\omega t \rangle = ½$

$\text{Power} = F_o\, \omega\, A\, (0 - ½ \sin\phi)$

and previously: $A = \dfrac{F_o / m}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2}}$ $\quad \sin\varphi = \dfrac{-\omega / \tau}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2}}$

$$Power = \frac{1}{2} F_o\, \omega \frac{F_o}{m} \frac{\omega / \tau}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2}$$

$$Power = \frac{1}{2} m \left(\frac{F_o}{m}\right)^2 \tau \frac{(\omega / \tau)^2}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2} = \frac{\frac{1}{2} m (\omega A)^2}{\tau}$$

Note the frequency dependence factor
containing ω, damping $\tau = c/m$, spring $\omega_o^2 = k/m$

$$factor = \frac{(\omega / \tau)^2}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\dfrac{\omega}{\tau}\right)^2}$$

C. Show that the **quality** of any oscillator $Q = \dfrac{\text{frequency at resonance}}{\text{full width at half maximum power}}$.

The quality factor of an oscillator is defined such that the energy loss in an oscillator is described exponentially with a time decay constant of τ = Q/ω.

$$Q = \frac{E/T}{P/2\pi} \quad with \quad \omega \equiv \frac{2\pi}{T} \quad and \quad P \equiv -\frac{\Delta E}{\Delta t} \quad then \quad \frac{\Delta E}{E} = -\frac{\Delta t}{\tau}$$

The quality of an oscillator is a figure of merit which transiently describes the number of repetitions (cycles) an oscillator will vibrate back and forth before coming to rest, after the driver energy source is removed i.e. before all its energy is dissipated via the oscillator's damping mechanism. The greater a quality factor, the longer the oscillator vibrates, or the less external power is required in order to keep the oscillator vibrating.

Thus $\dfrac{E}{E_o} = e^{-\frac{t}{\tau}} = e^{-\frac{\omega t}{Q}}$ $where\ Q \equiv \omega\tau$. The larger the ***Q***uality, the longer the energy lasts.

Power Absorption:

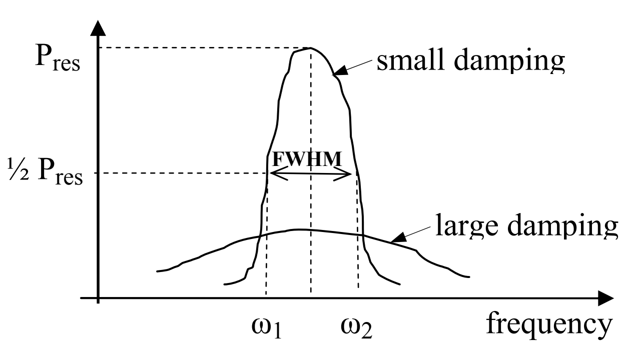
Power is the time average of the force times the velocity (see energy analysis):

$$Power = \langle F \cdot v \rangle = P_{res} \frac{(\omega/\tau)^2}{\left(\omega_o^2 - \omega^2\right)^2 + (\omega/\tau)^2} \qquad where \quad P_{res} = \frac{1}{2} m \left(\frac{F_o}{m}\right)^2 \tau$$

Near resonance ($\omega \approx \omega_o$) the half-power points occur when: $\omega_o^2 - \omega^2 = \omega/\tau$
or $2\,\omega_o\,\Delta\omega \approx \omega/\tau$

Thus the full width ($2\,\Delta\omega$) of the resonance at half maximum power is equal to $1/\tau$ and the quality of an oscillator ***Q*** measures the sharpness of damping near resonance:

$$Quality \quad Q = \omega\tau = \frac{\omega_o}{2\Delta\omega} = \frac{frequency\ at\ resonance}{full\ width\ at\ half\ maximum\ power}$$



where: $2\Delta\omega = \omega_2 - \omega_1$ is the "full width at half maximum power" of the oscillator's resonance curve, and ω_1 and ω_2 are the frequencies at which the power dissipation in the oscillator has dropped to one-half its resonance value:

$$Power_{resonance} = \frac{1}{2} m \left(\frac{F_o}{m}\right)^2 \tau$$

The quality factor is a measure of the sharpness of an oscillator's resonance curve.

Numerical Example of Damped Harmonic Oscillator

Given oscillator data:

mass	m = 1 gram
spring constant	k = 10 N/m
relaxation time	τ = 0.5 second

Then:

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{10\,\frac{N}{m}}{10^{-3}\,kg}} = 10^2\,\frac{radian}{second} = 16\,\frac{cycles}{second}$$

$$free\ oscillation\ frequency \quad \omega = \sqrt{\omega_o^2 - \left(\frac{1}{2\tau}\right)^2} = \sqrt{10^4 - 1} = 10^2\,\frac{radian}{second}$$

$$Quality\ factor \quad Q = \omega\tau = 10^2\,\frac{radian}{second}\,0.5\,second = 50$$

The ***amplitude*** dampens to e^{-1} of its initial value in a time: 2τ = 2 (0.5 sec) = 1 second.

If the damping is laminar, its damping constant is: c = m/τ = 1 gram/½sec) = 2 gram/second.

Given a driving force: F = 0.1 N sin (90 t) with amplitude F_o = 0.1 N ;

then $\frac{F_o}{m} = \frac{0.1\,N}{1\,gram} = 0.1\,\frac{N}{gram} = 100\,\frac{N}{kg}$ and the driving frequency is ω = 90 rad/sec.

The response of the oscillator is: φ =

$$Amplitude\ A = \frac{100\ N\ /\ kg}{\sqrt{\left(10^4 - 90^2\right)^2 + \left(\frac{90}{0.5}\right)^2}} = \frac{100}{1909} = 5\ cm \qquad \tan\varphi = \frac{90\ /\ 0.5}{90^2 - 10^4} = -0.095$$

$$the\ amplitude\ response\ at\ resonance\ is\ A(\omega = \omega_o) = \frac{F_o\ /\ m}{\omega_o\ /\ \tau} = \frac{100\ N\ /\ kg}{\frac{100\ rad\ /\ \sec}{0.5\ second}} = 50\ cm$$

$$the\ amplitude\ at\ low\ frequency\ is\ A(\omega = 0) = \frac{F_o}{k} = \frac{0.1\ N}{10\ N\ /\ m} = 1\ cm$$

The power absorption at resonance:

$$P_{res} = \frac{1}{2}\,10^{-3}\,kg\left(100\ N\ /\ kg\right)^2\ 0.5\sec = 2.5\,Watt$$

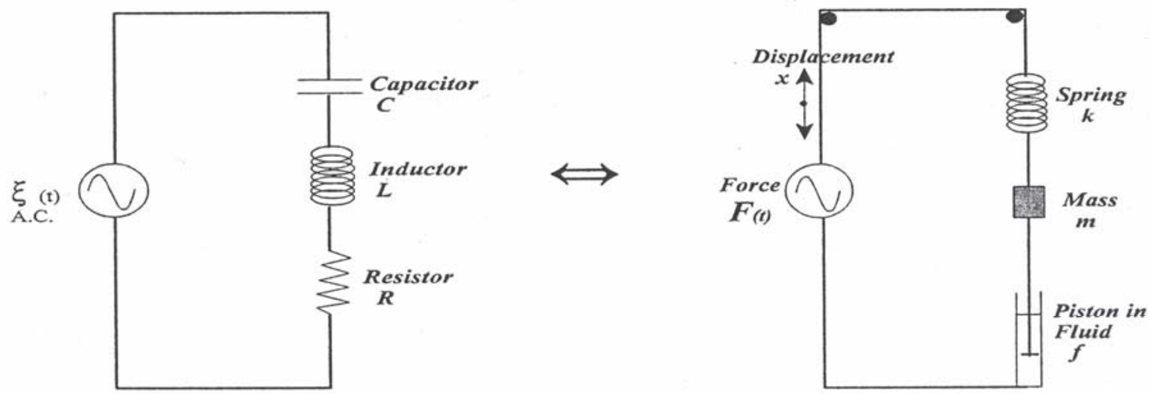
The full width of the resonance curve between half-power points is:

FWHM $\equiv 2\,\Delta\omega = \omega_o/Q$ = 100 rad/sec / 50 = 2 rad/sec = 0.32 Hz.

C. Deurzen

Comparison of Oscillators

Electrical | Mechanical



$$\xi_{(t)} = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

$$F_{(t)} = m\frac{d^2x}{dt^2} + f\frac{dx}{dt} + k\,x$$

The form of the electric circuit equation is identical to that of the mechanical, laminar-damped, driven forced oscillator. Mathematically, the two equations describe the same behavior. It remains to identify the quantities corresponding to each other and identify their physical significance and the role each plays.

Elastic restoration:	elastance $1/\boldsymbol{C}$	$\Longleftrightarrow$ $\boldsymbol{k}$ springiness
Inertial inactivity:	Inductance $\boldsymbol{L}$	$\Longleftrightarrow$ $\boldsymbol{m}$ mass
Energy dissipation:	Resistance $\boldsymbol{R}$	$\Longleftrightarrow$ $\boldsymbol{f}$ damping factor
External driver cause:	$\mathscr{E}$lectro Motive Force $\mathscr{E}_{(t)}$	$\Longleftrightarrow$ $\mathcal{F}_{(t)}$ motor
Response effect on system:	move charge $\boldsymbol{q}$	$\Longleftrightarrow$ $\boldsymbol{x}$ stretch-distance
Natural frequency $\omega_o^2 = \dfrac{elasticity}{inertia}$:	$\dfrac{1/C}{L}$	$\Leftrightarrow$ $\dfrac{k}{m}$

The inverse of capacitance plays the role of elasticity, as expected because a capacitor stores electric energy (charge effect) similar to a spring that stores elastic energy (stretch-distance effect).

Resistance plays the role of damping, as expected because it absorbs energy similar to a dashpot or a shock absorber in a mechanical system.

Inductance plays the role of inertia, as expected since it takes time to establish a current in an inductor when a voltage is applied to it.

Application: Nuclear (Mass) Energy

A nucleus can undergo transformation when some agent (nuclear) force changes the internal energy structure of the nucleus. During such a process, including fission or fusion, the system (nucleus) absorbs or throws off energy in the form of: interaction, particles, and/or electro-magnetic radiation energy. According to Einstein (experimentally verified) the existence of mass implies the presence of internal energy; mass is another form of energy.

For any (complex) system of mass *m*, its ***internal*** energy is $E = mc^2$ where c denotes the speed of light in vacuum (299 792 458 m/s). During a nuclear transformation, the change in internal energy due to a *mass* change is thus $\Delta E = \Delta mc^2$. The ***binding*** energy of a composite object of mass *m* against separation into parts is the mass difference multiplied by the speed of light squared: $\mathbf{J}_{\text{in or out}} = E_{binding} = \{ \text{(sum of masses of the separate parts)} - m \}\, c^2$.

Electromagnetic radiation is viewable either as waves or as energy packets called photons.
For a wave, its speed $c = f\lambda$ where f is the frequency and λ is the wavelength of the wave. Wave and photon are correlated by: a photon carries energy $E = hf$ (Einstein, 1905) and carries momentum $p = h/\lambda$ (DeBroglie, 1924) where h is Planck's (1900) constant.

<u>example</u>:

The fission of a uranium-235 atom throws off or releases 200 MeV of energy.
¿ What percent is this fission energy of the atom's total internal mass energy ?

Answer: $$\frac{\Delta E}{mc^2} = \frac{200\,MeV}{235\,amu}\left(\frac{1\,amu}{931\,MeV}\right) = 0.00091 \quad or \quad 0.091\,\%$$

¿ If some of the lost mass (energy) were detected in the form of electro-magnetic radiation having wavelength λ = 0.18 pm (gamma radiation), what is the energy of a detected photon ?

Answer: $$E = hf = \frac{hc}{\lambda} = \frac{1.2398\,eV\cdot\mu m}{\lambda} = \frac{1.2398\,eV\cdot\mu m}{0.18\,x\,10^{-12}\,m} = 6.89\,MeV\left(\frac{1.60\,x\,10^{-19}\,J}{1\,eV}\right) = 1.1\,x\,10^{-12}\,J$$

¿ If 75 % of a reaction's total released energy is deposited and harnessed into useful output energy, what is the power output of a nuclear plant burning 600 kg of uranium per year ?

Answer: $$Power = 600\,\frac{kg}{year}\left(\frac{1\,year}{3.16\,x\,10^{7}\,s}\right)0.00091\left(3.0\,x\,10^{8}\,\frac{m}{s}\right)^2 0.75 = 1.2\,x\,10^{9}\,\frac{kg\,\frac{m^2}{s^2}}{s} = 1.2\,GWatt$$
