Rotational Inertia

Consider a cart on four wheels (mass $m_1$) on a level track connected by a light string over a pulley to a second suspended mass $m_2$ (see diagram below). Assume both cart and pulley are frictionless, and ignore the mass of the string. Assume the string is non-stretchable, then both masses have the same acceleration. The tension force $F_{\text{tension}}$ in the string is the same on both sides of the pulley, because we are neglecting the friction in the pulley for the moment. Note: the suspended weight $m_2g$ is the applied force causing the entire system to accelerate. Analyze the motion of this system from the point of view using: (a) forces and (b) energy.

A. Force analysis of the motion of the system (cart plus suspended mass).

The free-body diagrams for the cart and suspended mass are illustrated to assist in analysis.

Combining these two equations by substituting $F_{\text{tension}}$ and accounting for the friction, results in:

$$m_2g - F_{\text{friction}} = (m_1 + m_2)a \quad \text{(1)}$$

Note, Equation (1) states: the net force (applied force $m_2g$ less the opposing friction) and the acceleration $a$ of the system are proportional with combined mass constant of $(m_1 + m_2)$. We may view the combined cart-plus-suspended mass as one system of total mass $m_1 + m_2$. In this analysis, the cause of any rotation of the pulley and the cart wheels has been ignored.
B. Energy analysis of the motion of the system (cart plus suspended mass).
As the system moves it gains kinetic energy of motion at the expense of the potential energy of position of the suspended weight $m_2g$. When the suspended weight falls through vertical distance $y$ its potential energy decreases by $m_2gy$, and using the conservation of energy:

$$m_2gy = KE_{\text{cart}}^{\text{translation}} + KE_{\text{rotation}}^{\text{pulley}} + KE_{\text{rotation}}^{\text{wheels}} + IE_{\text{friction}}$$

$$m_2gy = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}I_{\text{pulley}}\omega^2 + \frac{1}{2}I_{\text{wheels}}\omega^2 + RF_{\text{friction}}\theta$$

$R = \text{radius of pulley or wheels.}$

For non-slip rotational motion: $\omega = \frac{v}{R}$ and thus $I\omega^2 = \frac{I}{R^2}v^2$

$$m_2gy = \frac{1}{2}\left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{\text{pulley}} + 4\left(\frac{I}{R^2}\right)_{\text{wheel}}\right]v^2 + RF_{\text{friction}}\theta$$

We differentiate this energy equation with respect to time and recognize that:

$$\dot{v} = \frac{dv}{dt} = \text{system velocity} = v$$

$$\ddot{v} = \frac{d\dot{v}}{dt} = \text{system acceleration} = a$$

$$\dot{\theta} = \frac{d\theta}{dt} = \text{rotational velocity} = \omega = \frac{v}{R}$$

$$m_2g\ddot{y} = \frac{1}{2}\left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{\text{pulley}} + 4\left(\frac{I}{R^2}\right)_{\text{wheel}}\right]2v\dot{v} + RF_{\text{friction}}\dot{\theta}$$

$$m_2g = \left[m_1 + m_2 + \left(\frac{I}{R^2}\right)_{\text{pulley}} + 4\left(\frac{I}{R^2}\right)_{\text{wheel}}\right]a + F_{\text{friction}} \quad (2)$$

Note: Equation $(2)$ is the same as $(1)$ stating force is proportional to acceleration but with a different mass constant which incorporates the rotational inertia of the pulley and wheels.

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